* 1. Introduction and Well-Ordering

A fundamental property of the natural numbers, i.e.. the positive integers **N** = {1,2,3…,}, that will be ued throughout the book is the fact that they are well ordered. This means that any nonempty subset has a smallest element Smin such that Smin s for all s . Using the natural ordering of the integers, rational numbers, or real numbers, we see that this property does not hold for those numbers.

* 1. Elementary Linear Algbera

Vectors create different shapes

* 1. fields

The “scalars” or numbers used in linear algebra all lie in a field. A field is a set F of numbers, where one has both addition

F x F F

( +

And multiplication

F x F F

(

Definition 1.3.1 A field F is a set whose elements are called numbers or when used in linear algebra scalars. The field contains two different elements 0 and 1, and we can add and multiply numbers. These operations satisfy

1. The associative law

+ +­)+

1. The commutative law

+ *=*

1. Addition by 0:

1. Existence of negative numbers: For each we can find - so that

+(- *) = 0*

1. The associative law

1. The commutative law:
2. Multiplication by 1:
3. Existence of inverses: For each we can find ­-1 so that

-1=1

1. The distributive law:

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De morgans law negates the value of different variables

* 1. Vector Spaces:

Definition 1.4.1 A vector space consists of a set of vectors V and a field F. The vectors can be added to yield another vector: if x, y then x + y V or

V x V -> V

(x, y) -> x + y

Proposition 1.4.3 Let V be a vector space over a field F. If x and then:

1. 0x = 0
2. -1x = -x
3. If , then either

DEF 1.4.9 A nonempty subset M Be a subspace if it is closed Under addition And scalar multiplication

X, y, ⇨ x+y

* 1. Bases

Def 1.5.1 Our first construction is to form linear combinations of vectors. If1…m F and x1…xm V, then we can multiply xi by the scalar I and then add up the resulting vectors to form the linear combination:

X = 1x1+…m xm

Def 1.5.2 A finite basis for V is a finite collection of vectors x1 … xn V such that each element x V can be written as a linear combination.

X = 1x1 + … + nxn

Def 1.5.3 This allows us to define the dimension of V over F to be the number of elements in a basis. Note that the uniqueness condition for the linear combinations guarantees that none of the vectors in a basis can be zero vector.

Lemma 1.5.5 Let V be a vector space over F. If V has a basis with one element then any other finite basis also has one element.

* 1. Linear Maps

Def 1.6.1 A map L: V -> W between vector spaces over the same field F is said to be linear if it preserves scalar multiplication and addition in the following way:

L =(x) = L(x)

L(x + y) = L(x) + L(y)

Where F and x, y V

It is possible to collect these two properties into one condition as follows

L(1x1 +2x2) = 1L(x1) + 2L(x2)

Where 1, 2 F and x1 , x2 V.

Def 1.6.2 If the values are the field itself, i.e. W = F, then we also call L : V -> F a linear function or linear functional. If V = W, then we call L: V -> V a linear operator

Def 1.6.3: The set of all linear maps L: V -> W is often denoted Hom(V, W). In case we need to specify the scalars, we add the field as a subscript HomF(V, W)

The abbreviation Hom stands for homomorphism. Think abstract

Def 1.6.4 Let L : V -> W be a linear operator. A subspace M V is said to be L-invarient or simply invariant if L(M) M.

Examples for 1.6.4-1.6.10

Def 1.6.10 The trace is a lienar map on swuare matrices that adds the diagonal entries

Tr : MatnXn (F) -> F

Tr(A) =11+22 + … + nn

Lemma 1.6.11 (Invariance of Trace) If A MatmXn (F) and B MatnXm(F) then A B MatmXm (F), BA MatnXn (F) and tr(AB) = tr(BA)

Corrollary 1.6.12 There are no matrices A, B MatnXn (F) such that

AB – BA = 1Fn

1.7 Linear Maps as Matrices

Lemma 1.7.1 Assume V is one-dimensional over F, then any L : V -> V is of the form L = lV .

Lemma 1.7.2 Any linear map L: Fm -> V of the form

L = [x1…xm]

Where xi = L (ei)

* 1. Dimension and Isomorphism

Def 1.8.1 We say that a linear map L : V -> W is an isomorphism if we can find K : W -> V such that LK = Lw and KL = lv

Def 1.8.2 Two vector spaces V and W over F are said to be isomorphic if we can find an isomorphism L : V -> W

Lemma 1.8.3 V and W are isomorphic if and only if there is a bijective linear map L : V -> W

The “if and only if” part asserts that the two statements :

* V and W are isomorphic
* There is a bijective linear map : V -> W

Are equivalent. In other words, if one statement is true, so is the other. To establish the propistion it is therefore necessary to prove two things, namely that the first statement implies the second and that the second implies the first.

Th 1.8.4 (Uniqueness of Dimension) If Fm and Fn are isomorphic over F then n = m

Def 1.8.5 We can now unequivocally denote and define the dimension of a vector space V over F as dimF V = n if V is isomorphic to Fn . In case V is not isomorphic to any Fn , we say that V is infinite dimensional and write dimF V =

Corollary 1.8.6 If V and W are finite dimensional vector spaces over F, then HomF (V, W) is also finite dimensional and

DimF HomF (V, W) = (dimF ­W) \* (dimF V)

* 1. Matrix Representations Revisted

Demo on the board

* 1. Subspaces

Def 1.10.6 If M, N V are subspaces, then we can form two new subspaces, the sum and the intersection:

M + N = {x + y : x M and y N }

M N = { x : x M and x N }

Def 1.10.7 If S V is a subset of vector space, then the span of S is defined as

Span (S) =

Where M V is always a subspace of V. Thus the span is the intersection of all subspaces that contain S. This is a subspace of V and must in fact be the smallest subspace containing S

Proposition 1.10.8 Let V be a vector space and S, T, V subsets

1. If S T, then span (S) span (T)
2. If M V is a subspace, then span (M) = M
3. Span (span (S)) = span (S)
4. Span(S) = span(T) if and only if S

Lemma 1.10.9 (Characterization of span (M)) let S Then, span (S) consist of all linear combinations of vectors in S.

Def 1.10.11 If we combine the two concepts of transversality and trivial intersection, we arrive at another important idea. Two subspaces are said to be complementary if they are transversal and have trivial intersection

Lemma 1.10.12 Two subspaces M, N V are complementary if and only if each vector z V can be written as z = x + y, where x M and y in one and only one way

Def 1.10.13 When we have two complementary subspaces M, NV we also say that V is a direct sum of M and N and we write this symbolically as V = M ⊕ N. The special sum symbol indicates that indeed, V = M + N and also that the two subspaces have trivial intersection. Using what we have learned so far about subspaces, we get a result that is often quite useful

Corollary 1.10.14 Let M, N V be subspaces. If M N = {0}, then

M + N = M ⊕ N

And if both are finite dimensional then

Dim (M + N) = dim(M) + dim(N)

Th 1.10.20 (Existence of Complements) Let M V be a subspace and assume that V = span {x1… xn}. If M V, then it is possible to choose xil … xik {xl … xn} such that

V = M ⊕ span (xil … xik}

* 1. Linear Maps and Subspaces

Def 1.11.1 Let L : V -> W be a lienar map between vector spaces. The kernal or nullspace of L is

Ker(L) = N(L) = L-1 (0) = {x V : L(x) = 0|

The image or range of L is

im(L) = R(L) = L(V) = {y L(x) for some x V }

Lemma 1.11.2 ker(L) is a subspace of V and im(L) is a subspace of W.

Lemma 1.11.3 L is one to one if and only if ker(L) = {0}

Def 1.11.4 The map E : V -> V is defined as follows. For z V, we write z = x + y for unique x M, yN and define

E(z) = x

Thus, im(E) = M and ker(E) = n

Lemma 1.11.5 (Uniqueness of Complements) if V = Mi ⊕ N = M2 ⊕ N, then M1 and M2 are isomorphic

Th.1.11.6 (The subspace theorem) assume that V is finite dimensional and that M⊂ V is a subspace. Then M is finite dimensional and

DimF M DimF V

Moreover if V = M ⊕ N then

DimF  V = dimF M + dimF N

Th. 1.11.7 (The dimension formula) Let V be finite dimensional and L: V -> W a linear map, then im(L) is finite dimensional and

DimF V = dimF ker(L) + dimF im(L)

Def 1.11.8 The number nullity (L) = dimF ker (L) is called the nullity of L and rank(L) = dimF  im(L) is known as the rank of L.

Corollary 1.11.9 If M is a subspace of V and dimF M = dimF V = n < , then M = V

Corollary 1.11.10 Assume that L : V -> W and dimF V = dimF W < . Then, L is an isomorphism if either nullity (L) = 0 or rank(L) = dim W.

Corollary 1.11.11 If L : V -> W is a linear map between finite dimensional spaces, then we can find bases e1… em for V and f1 … fn for W so that

L(e1) = f1

L(ek) = fk

L(ek+1) = 0

L(em) = 0

Where k = rank(L)

Th.1.11.12 (Characterization of projections) projections all satisfy the functional relationship E2 = E. Conversely any E : V -> V that satisfies E2 = E is a projection

* 1. Linear Independence

Def 1.12.1 Let x1 … xm be vectors in a vector space V. We say that x1 … xm are linearly independent if

X11+ … + xm m = 0

Impies that 1 = … = m = 0

Def 1.12.2 Conversely, we say that x1 … xm are linearly dependent if they are not linearly independent i.e.. we can find 1 … m  F not all zero so that x11 +…+ xm­m = 0

Lemma 1.12.3 (Characterization of linear dependence) Let x1 … xn V. Then x1 … xn are linearly dependent if and only if either x1 = 0 or we can find a smallest k2 such that xk is a linear combination of x1…xk-1.

Corollary 1.12.4 (Characterization of linear independence) Let x1 … xn V. Then x1…xn are linearly independent if and only if x1 0 and for each k2

Xk